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Invariant States of a Thermally Conducting Barrier

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For an infinite one-dimensional system representing a thermally conducting barrier and two semi-infinite reservoirs which it separates, we prove the existence of a unique stationary probability distribution, to which essentially any initial distribution converges for large times.

KEY WORDS: Invariant states; thermally conducting barrier.

1. INTRODUCTION

A central problem in nonequilibrium statistical mechanics concerns the microscopic description of nonequilibrium steady-state phenomena such as heat flow or fluid flow. The ultimate goal is, perhaps, a formula, à la Gibbs, for the measures providing such a description. At present, such a goal is quite distant. Indeed, in most cases, for example that of fluid flow past a fixed obstacle, and setting aside questions of uniqueness and formula, it is not even clear mathematically that a stationary probability distribution on microscopic states exists. (On the level of the Boltzmann equation, the existence of stationary distributions for flow past an obstacle has recently been established.[(1)])

We here consider a simple model: a one-dimensional infinite system representing a region Λ and two semi-infinite reservoirs separated by Λ. Each reservoir contains an "ideal gas" of atoms which flow into Λ and

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interact there with molecules confined inside Λ. The equilibrium case in which the reservoirs have the same temperature and density was studied in Ref. 2. We are concerned here with the nonequilibrium situation in which the left reservoir is described by different parameters from the right—for example, the temperature or density (or both) may differ. To every choice of the density and velocity distribution of incoming particles for the two reservoirs, we prove that there corresponds a unique stationary probability measure for the entire infinite system (provided zero-velocity particles are not allowed). At the same time, we show that the system in Λ has a unique stationary probability distribution, to which every initial distribution in Λ converges as the time $t \to \infty$ (convergence to the nonequilibrium steady state).

We proceed as follows: In Section 2 the model is described in more detail, with particular emphasis on the Markov process describing the evolution in Λ. In Section 3, the key technical estimates on the distribution in Λ at time t, starting from an arbitrary initial distribution without zero velocity particles, are obtained. These estimates, uniform at t, are used in Section 4 to prove the main results: Theorem 4.1, Corollary 4.3, Theorem 4.5, and Remark 4.6.

2. THE MODEL

We consider a one-dimensional system consisting of a region $\Lambda = [-L/2, L/2]$ of length L separating two reservoirs. There are two species of particles: a finite number J of *molecules* which are confined to Λ, undergoing elastic collisions at its walls, and *atoms* which pass freely through the walls of Λ. The molecules and atoms have the same mass, all collisions between particles are elastic, and no other forces act. Since an atom trapped between molecules will behave like a molecule, we assume that the molecules are adjacent.

The state space X within Λ is a union of n-particle state spaces:

$$X = \bigcup_{n=J}^{\infty} X_n$$

Explicitly $X_n = I_n \times Q_n \times V^n$, where V is the one-particle velocity space, Q_n is position space,

$$Q_n = \{(q_1, \ldots, q_n| - L/2 \leqslant q_1 \leqslant q_2 \cdots \leqslant q_n \leqslant L/2\}$$

and $I_n = \{1, 2, \ldots, (n - J + 1)\}$; the index $i \in I_n$ specifies which particle in Λ is the left-most molecule. We take $V = \mathbb{R}\backslash\{0\}$ for reasons discussed in Remark 2.4 below.

The regions $\Lambda_{\pm} = \{q | \pm q > L/2\}$ are reservoirs; we consider only reservoir atoms which are initially moving toward Λ. Let Ω denote the space of all initial reservoir states, with **P** the probability measure on Ω. **P** is specified as follows: at time $t = 0$ the particles in $\Lambda_{\pm}$ are Poisson distributed in space with density $\rho_{\pm}$; their velocities v satisfy $\mp v > 0$ and their speeds $|v|$ are independent of all positions and all other speeds and have distributions $\pi_{\pm}$, where $\pi_{\pm}$ are probability measures on $\{v \mid v > 0\}$. We require that $\pi_{\pm}$ correspond to a finite flux of particles

$$f_{\pm} = \rho_{\pm} \int_0^{\infty} v \, d\pi_{\pm} < \infty \tag{2.1}$$

for definiteness we assume

$$\rho_{+} > 0 \tag{2.2}$$

and

$$f_{+} \geqslant f_{-} \tag{2.3}$$

In the case $f_{+} = f_{-}$ we need an additional technical restriction on the number of slow particles in the reservoirs:

$$\int_0^{\infty} v^{-1/2} \, d\pi_{\pm} < \infty \tag{2.4}$$

[The necessity of (2.4) is presumably an artifact of our proof.]

Consider now the dynamics of the model. For every initial state $y \in X$, initial reservoir configuration ω, and time $t \geqslant 0$, we wish to have a well-defined state $x_t \equiv x(y, \omega, t) \in X$. There are three difficulties:

(i) Certain ω lead to an infinite number of particles entering Λ in finite time. The set of such ω has **P** measure 0, however, so we may exclude them from Ω.

(ii) The state space X we have defined is incorrect physically; the state at the moment of a collision is not unique without tedious identifications in X. We remedy this by assuming that $x(y, \omega, t)$ is right continuous in t for fixed y, ω, and we remove from X all points which are not points of right continuity for some $x(y, \omega, t)$.

(iii) The dynamics as described above do not determine the outcome of multiple collisions. We prescribe the behavior in such cases as follows: if the collision occurs at time t, we perturb the particle positions (but not velocities) just before t, so as to remove all degeneracy, find the state immediately after t, and take its limit as the perturbation is scaled to zero. (If several particles enter the collision on the same trajectory, the perturbation must respect the ordering of atoms and molecules.) In most cases the result is independent of the perturbation chosen, but this is not true when the collision involves a wall, one or more atoms, and one or more molecules. In this case we specify the perturbation explicitly and rather arbitrarily: for small $\epsilon > 0$, the perturbed trajectory of a particle with speed v

which is in Λ before the collision will strike the wall at time $t - \epsilon v$; remaining degeneracy does not require explicit treatment.

Conditions (i)–(iii) specify $x(y, \omega, t)$ completely.

The well-defined dynamics and the measure $\mathbf{P}$ on Ω yield a Markov process $P^t(y \mid dx)$ on X. P^t acts on Borel measures by

$$\mu P^t(dx) = \int P^T(y \mid dx)\, \mu(dy)$$

and on Borel functions by

$$(P^t f)(y) = \int_X P^t(y \mid dx)\, f(x)$$

The main result of this paper is the existence of a unique invariant measure ν for this process such that, for any μ, $\| \mu P^t - \nu \| \to 0$ as $t \to \infty$, where $\| \ \|$ is the variation norm in measures.

We close this section with a series of remarks which will be needed later in the paper.

Remark 2.1. The equality of the particle masses implies an alternate way to view the dynamics of our system. We may consider the particles as noninteracting except that (i) all particles are labeled as "atom" or "molecule"; when two particles pass each other, they exchange labels, and (ii) particles carrying the molecule label are reflected elastically at the walls of Λ. The trajectory of such a noninteracting particle, inside Λ or the reservoirs, will be called a *pulse*.

Remark 2.2. We fix an initial measure μ on X and introduce a basic decomposition of the measure μP^t. Let N_t denote the random variable on $X \times \Omega$ whose value at (y, ω) is the number of pulses (or particles) in Λ at time t, given initial state $y \in X$ and $\omega \in \Omega$. We write $N_t = N_{at} + N_{bt}$, where N_{at} (respectively, N_{bt}) counts the number of *pulses* present in Λ at time t which entered Λ from the reservoirs at some time $s > 0$ (respectively, which have persisted since time $t = 0$). Finally, for a Borel set $B \subset X$, define

$$\mu_a^t(B) = (\mu \times \mathbf{P})\big[\{x_t \in B \text{ and } N_{bt} = 0\}\big]$$

$$\mu_b^t(B) = (\mu \times \mathbf{P})\big[\{x_t \in B \text{ and } N_{bt} = 0\}\big]$$

so that $\mu P^t = \mu_a^t = \mu_b^t$. That is, μ_a^t is the probability distribution on states of the system at time t arising from system histories for which all initial pulses have exited from Λ.

Definition 2.3. We define two families of finite reference measures, one using only the velocities from Λ_+ [which, by (2.2), *must* occur in any invariant state], the second using velocities from both reservoirs and, in the case $f_+ = f_-$, with an enhanced probability of low-velocity particles:

(i) the measure π^* and π on V, defined by

$$d\pi^*(v) = \rho_+ \, d\pi_+(|v|)$$
$$d\hat{\pi}(v) = \left[\rho_+ \, d\pi_+(|v|) + \rho_- \, d\pi_-(|v|)\right]$$
$$d\pi(v) = g(|v|)\, d\hat{\pi}(v)$$

where for $w > 0$

$$g(w) = \begin{cases} w + 1 & \text{if } \; f_+ \neq f_- \\ w + |w|^{-1/2} & \text{if } \; f_+ = f_- \end{cases}$$

(ii) the measures λ_n^* and λ_n on $Q_n \times V^n$, defined as the product of Lebesgue measure on Q_n and $(\pi^*)^n$ or $(\pi)^n$ on V^n;

(iii) the measures λ^* and λ on X, defined by $\lambda^*|_{\{i\}|\times Q_n \times V^n} = \lambda_n^*$, $\lambda|_{\{i\}\times Q_n \times V^n} = \lambda_n$, for $1 \leqslant i \leqslant n - J + 1$.

Remark 2.4. We may vary our treatment of zero-velocity particles by choice of the velocity space V. Since the number of such particles in Λ, and their positions, are constants of the motion, we expect the invariant measures on the state space defined with $V = \mathbb{R}$ to be indexed by the set of probability measures on $\bigcup_{n \geqslant 0} Q_n$ describing the locations of zero-velocity particles. In fact, this can easily be verified by the methods of this paper. For simplicity, we study in detail only the case $V = \mathbb{R}\backslash\{0\}$ where no such particles are present.

3. ESTIMATES ON THE MEASURES μP^t

Throughout this section we take μ to be a fixed initial measure on X and $\mathbf{Q}$ the measure $\mu \times \mathbf{P}$ on $X \times \Omega$. The discussion of the invariant measure in Section 4 requires two key estimates on the measure μP^t, which we develop here. The first of these (Theorem 3.5) is essentially a *tightness* condition,[3] and shows that neither a large number of particles nor arbitrarily slow particles tend to accumulate in Λ. The second (Theorem 3.6) is an absolute continuity condition which shows similarly that particles do not tend to accumulate in increasingly small regions of X. Both these estimates come from the fundamental Lemma 3.1, which enables us to bound the probability that a given pulse persists in Λ through several reflections.

To understand the following notations, used in the proof of Lemma 3.1, it helps to visualize the process by identifying pulses with their trajectories in space-time $\mathbb{R} \times \mathbb{R}_+ \supset \Lambda \times \mathbb{R}_+$, as in Fig. 1. Given times $0 \leqslant T_1 < T_2$, we let $n_\pm(T_1, T_2)$ denote the number of pulses entering Λ from $\Lambda_\pm$ during the interval $[T_1, T_2]$. For $t \in [T_1, T_2]$, let $m_t[T_1, T_2]$ denote the number of pulses which (i) intersect $\Lambda \times [T_1, T_2]$ and (ii) at time t, lie to

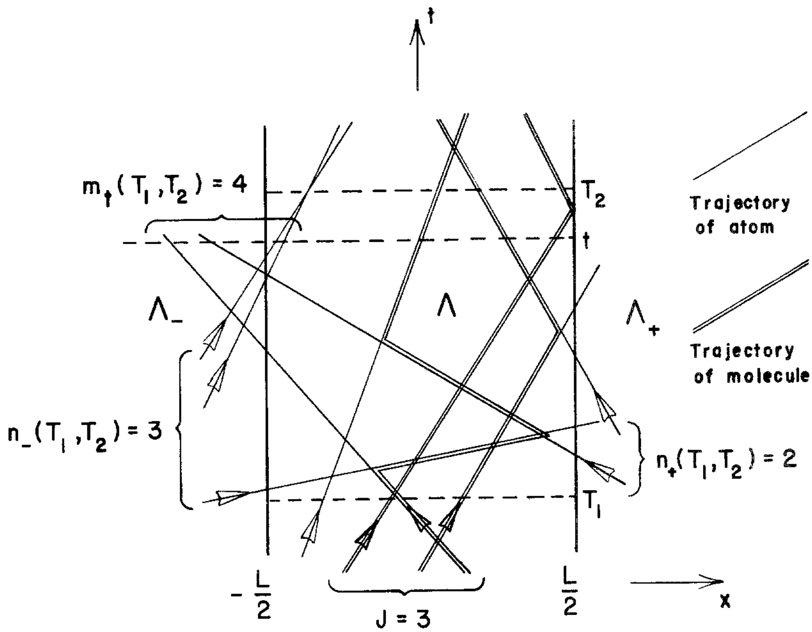

Fig. 1. Interpretation of J, m_t, n_+, and n_-.

the left of the left-most molecule. The dynamics imply that $m_t[T_1, T_2]$ is independent of t; in particular,

$$m_{T_1}[T_1, T_2] = m_{T_2}[T_1, T_2] \tag{3.1}$$

Finally, let $\mathscr{A}[T_1, T_2]$ denote the σ algebra on Ω (or on $X \times \Omega$) generated by the velocities and entry times of particles entering Λ during $[T_1, T_2]$.

Lemma 3.1. For $0 \leqslant T_1 < T_2$, let $F \subset X \times \Omega$ be the event that the left-most molecule is at the left wall at time T_1 and the right-most molecule at the right wall at time T_2. Then there is an event $F_0 \in \mathscr{A}[T_1, T_2]$ with $F \subset F_0$ and with $\mathbf{Q}(F_0) \equiv \mathbf{P}(F_0) \equiv \eta$ depending only on $T = T_2 - T_1$ and satisfying

(i) $k_0 \leqslant \eta(T) < 1$ for some $k_0 > 0$ and all $T > 0$;

(ii) there are constants $k_1, k_2 > 0$ such that for sufficiently large T,

$$1 - \eta(T) > \begin{cases} k_1 & \text{if } f_+ > f_- \\ k_2 T^{-1/2} & \text{if } f_+ = f_- \end{cases}$$

Proof. Let $F_1 \in \mathscr{A}[T_1, T_2]$ be the event that $n_+[T_1, T_2] \geqslant J$ and that the last J pulses to enter Λ in $[T_1, T_2]$ do so from Λ_+, say at times

$t_1 \leqslant \cdots \leqslant t_J < T_2$, and with speeds $|v_j| < L(T_2 - t_j)^{-1}$. Thus on $F \cap F_1$, these J pulses lie in Λ to the left of the right-most molecule at time T_2, so that

$$m_{T_2}[t, T_2] \geqslant 1 \tag{3.2}$$

for any $t < T_2$. Certainly $\mathbf{P}(F_1) \equiv h_1$ depends only on T and $h_1(T)$ is a strictly increasing function.

Now let $t_0 \in [T_1, T_2]$ be the last collision time of a molecule with the left wall. Using (3.1) we have

$$n_-[t_0, T_2] = m_{t_0}[t_0, T_2] = m_{T_2}[t_0, T_2] \tag{3.3}$$

If $t_0 \geqslant t_1$ then $n_-[t_0, T_2] = 0$, contradicting (3.2), (3.3), so necessarily $t_0 < t_1$ on $F \cap F_1$. But then no pulses entering Λ from Λ_+ after t_0 may be reflected from the left wall before T_2, so that on $F \cap F_1$,

$$m_{T_2}[t_0, T_2] \geqslant n_+[t_0, T_2] - J + 1 \tag{3.4}$$

Now consider a stochastic process

$$M(s) = n_+[T_2 - s, T_2] - n_-[T_2 - s, T_2]$$

defined on $(\Omega, \mathbf{P})$ for $0 \leqslant s \leqslant T$. M describes a random walk on a one-dimensional lattice with exponentially distributed jumping times, with rates f_+ and f_- for jumps to the right and left, respectively. Observe that on F_1, $M(T_2 - t_1) = J$, while on $F \cap F_1$, $M(T_2 - t_0) \leqslant J - 1$ from (3.3), (3.4). Hence $F \cap F_1 \subset F_2$, where $F_2 \in \mathscr{A}[T_1, T_2]$, $F_2 = F_1 \cap \{\omega \mid M(s) - M(T_2 - t_1) = -1$ for some s, $T \geqslant s > T_2 - t_1\}$. But on F_1, $T_2 - t_1$ is a stopping time for $M(s)$, so $\mathbf{P}(F_2 \mid F_1)$ is bounded by the probability of a passage of $M(s)$ through $M(s) = -1$ in time T. From Ref. 4, Vol. I, p. 272 and Vol. II, p. 60, $\mathbf{P}(F_2 \mid F_1) \leqslant 1 - h_2(T)$, where $h_2(T) > 0$ for all $T > 0$ and

$$h_2(T) > \begin{cases} 1 - \dfrac{f_-}{f_+} & \text{if } f_- < f_+ \\ kT^{-1/2} & \text{for some } k \text{ and large } T, \quad \text{if } f_- = f_+ \end{cases} \tag{3.5}$$

Finally, set $F_0 = (\Omega \backslash F_1) \cup F_2$; then $F \subset F_0$ and

$$\begin{aligned} \eta(T) \equiv \mathbf{P}(F_0) &= 1 - \mathbf{P}(F_1) + \mathbf{P}(F_1)\mathbf{P}(F_2 \mid F_1) \\ &= 1 - h_1(T)h_2(T) \end{aligned}$$

Then $\eta(T) < 1$ since $h_1(T), h_2(T) > 0$ for $T > 0$, and $\eta(T) \geqslant \inf_T \mathbf{P}(F_0) \equiv k_0$, easily seen to be strictly positive. The estimate (ii) follows from (3.5) and the monotonicity of h_1.

In stating our next result it is convenient to fix a speed $v^* > 0$ which serves to distinguish slow from fast particles.

Corollary 3.2. A pulse with speed v present in Λ at time t_1 can remain in Λ until time t_2 only if there occurs a certain event $F \in \mathscr{A}(t_1, t_2)$ which satisfies

$$\mathbf{P}(F) \leqslant k_0^{-2}\theta(v)^{v(t_2-t_1)/2L} \tag{3.6}$$

where k_0 is as in Lemma 3.1 and

$$\theta(v) = \begin{cases} \eta(L/v) & \text{if} \quad v \leqslant v^* \quad (3.7\text{a}) \\ \beta^{v^*/v} & \text{if} \quad v > v^* \quad (3.7\text{b}) \end{cases}$$

for some β, $0 < \beta < 1$.

Proof. Let r be the maximal integer such that $r \leqslant v(t_2 - t_1)/2L$. Apply Lemma 3.1 to each of the $r - 1$ passages from left to right that the pulse must make to survive until t_2, and let F be the intersection of the events F_0 thus obtained. Then

$$\mathbf{P}(F) = \eta(L/v)^{r-1}$$

and (3.6), (3.7a) follow (for all v). This bound is not useful for large v because $\eta(T) \to 1$ as $T \to 0$. Hence for $v > v^*$, let $p \geqslant 2$ be the minimal integer so that $w \equiv v/p \leqslant v^*$ and let r be the maximal integer such that $r \leqslant w(t_2 - t_1)/2L$. We obtain F by applying the argument above to the $r - 1$ disjoint time intervals of length $(2p - 1)L/pw$ in each of which the pulse makes p left to right trips. This yields

$$\mathbf{P}(F) = \eta\left(\frac{(2p-1)L}{pw}\right)^{r-1}$$

from which (3.7b) follows with

$$\beta = \sup_{1.5 \leqslant \alpha \leqslant 3} \eta\left(\frac{\alpha L}{v^*}\right)^{1/2}$$

Lack of uniformity in our estimates of the behavior of μP^t can arise for a large number of particles or, in some cases, for slow particles. We therefore say that a set $K \subset X$ is *bounded* if (i) $K \subset \bigcup_{J \leqslant n \leqslant N} X_n$ for some N, and (ii) there is a minimum speed $w > 0$ such that $|v_j| > w$ for all $x = (i, q_1, \ldots, q_n, v_1, \ldots, v_n) \in K$ and all j, $1 \leqslant j \leqslant n$.

Corollary 3.3. For any initial μ, let μ_b^t be defined as in Remark 2.2. Then $\mu_b^t \to 0$ in variation norm as $t \to \infty$.

Proof. We may find a bounded set $K \subset X$ such that $\mu(X \setminus K)$, and hence also $\|(\mu|_{X\setminus K})_b^t\| \leqslant \|\mu|_{X\setminus K} P^t\|$, is arbitrarily small. Thus it suffices to prove the corollary with μ replaced by $\tilde{\mu} = \mu|_K$. Suppose $K \subset \bigcup_{n \leqslant N} X_n$

with minimum speed w, as above. Now $\| \tilde{\mu}_b^t \|$ is the probability that at least one of the initial pulses survives until t; by Corollary 3.2,

$$\| \tilde{\mu}_b^t \| \leqslant N k_0^{-2} \Big[\sup_{v \geqslant w} \theta(v)^v \Big]^{t/2L}$$

We next estimate the correlation functions of particles which enter Λ after $t = 0$. For this purpose we extend the notation of Remark 2.2: if $A \subset Q_n \times V^n$ for $n \geqslant 1$, and $x = (i, q_1, \ldots, q_m, v_1, \ldots, v_m)$ is the state of the system at time t, we let $N_{at}(A)$ denote the number of n-tuples of distinct indices $(i_1, \ldots, i_n)$ such that (i) $(q_{i_1}, \ldots, q_{i_n}, -v_{i_1}, \ldots, v_{i_n}) \in A$ and (ii) the i_jth pulse entered Λ after time $t = 0, j = 1, \ldots, n$.

Lemma 3.4. For each $n \geqslant J$ there exists a constant c_n such that for $A \subset Q_n \times V^n$,

$$E(N_{at}(A)) \leqslant c_n \lambda_n(A)$$

Here E is the expectation with respect to $\mathbf{Q}$.

Proof. We give a somewhat informal proof which can easily be formalized, e.g., by a slight extension of Palm measure.[5] Thus for given $q_1, \ldots, q_n$, and $v_1, \ldots, v_n$, let $dA \subset X_n$ denote the set

$$\prod_j [q_j, q_j + dq_j] \times \prod_j [v_j, v_j + dv_j]$$

We show that

$$E(N_{at}(dA)) \leqslant c_n \prod_j dq_j \pi(dv_j) = c_n \lambda_n(dA) \tag{3.8}$$

where $\pi(dv_j) = \pi([v_j, v_j + dv_j])$ with π as in Definition 2.3.

Consider any system history which contributes to $E(N_{at}(dA))$, and suppose that in this history the pulse which is at (q_j, v_j) at time t has been reflected k_j times from the walls at $\pm L/2$. Then this pulse must have entered Λ at time $t_j \equiv t^{k_j}(q_j, v_j)$, where

$$t^k(q, v) = t - q/v - (2k+1)L/2|v|$$

more precisely, it must have entered with velocity v in an interval $[\pm v_j, \pm v_j \pm dv_j]$ and at some time between $t^{k_j}(q_j, v)$ and $t^{k_j}(q_j + dq_j, v)$. Let G_j be the event that such a pulse enters; the *a priori* probability of G_j is bounded by $dq_j \hat{\pi}(dv_j)$ (see Definition 2.3).

We assume for notational convenience that $t_1 < t_2 < \cdots < t_n$; in fact our estimate is independent of ordering. Let $I_j = [a_j, b_j]$, where $a_j = t_j + (j-1)(t - t_j)/n$ and $b_j = a_j + (t - t_j)/n$, and suppose that the points t_l with $l > j$ partition I_j into subintervals $I_{j1}, \ldots, I_{jm_j}$. We apply Corollary 3.2 in I_{jr} to the pulse which entered at t_j, to produce an event F_{jr}; all the events F_{jr} and G_j are independent (we treat dq_j as infinitesimal). Then

$$\begin{aligned} E(N_{at}(dA)) &\leqslant \sum_{k_1, \ldots, k_n = 0}^{\infty} \mathbf{P}\left[\bigcap_{j=1}^{n} \left(\bigcap_{r=1}^{m_j} F_{jr} \right) \cap G_j \right] \\ &\leqslant \sum_{k_1, \ldots, k_n} \prod_j \left(\prod_r k_0^{-2} \theta(|v_j|)^{|v_j|\,|I_{rj}|/2L} \right) dq_j \, \hat{\pi}(dv_j) \\ &\leqslant k_0^{-(4n-2)} \prod_j \left(\sum_{k=0}^{\infty} \theta(v_j)^{k/2n} \right) dq_j \, \hat{\pi}(dv_j) \\ &\leqslant c_n \prod_j dq_j \, \pi(dv_j) \end{aligned}$$

Here $|I|$ denotes the length of the interval I and we have used the easily derived formulas

$$\sum_{j=1}^{n} m_j \leqslant 2n - 1$$

$$\sum_r |I_{jr}| = |I_j| = \frac{t - t_j}{n} \geqslant \frac{k_j L}{n|v_j|}$$

$$\frac{1}{1 - \theta(v)^a} \leqslant (\text{const}) g(v) \qquad \text{for} \quad a > 0$$

Theorem 3.5. For any initial measure μ and $\epsilon > 0$ there exists a bounded set $K \subset X$ with $\mu P^t(X \setminus K) < \epsilon$ for all $t \geqslant 0$.

Proof. As in the proof of Corollary 3.3 we may reduce to the case in which μ has support in a bounded set $K_0 \subset X$. Suppose $K_0 \subset \bigcup_{J \leqslant n \leqslant N_1} X_n$ and that pulses in K_0 have minimum speed $w > 0$. By taking $n = 1$ and $A = \Lambda \times V$ in Lemma 3.4 we see that $E(N_{at})$ is bounded uniformly in t, so that there exists N_2 with $\mathbf{Q}[N_{at} \geqslant N_2] < \epsilon/2$; hence if $N = N_1 + N_2$,

$$\mu P^t\left(\bigcup_{n \geqslant N} X_n \right) < \epsilon/2 \tag{3.8}$$

By Lemma 3.4, we may choose u with $0 < u < w$ such that, if K is the event that there are at most N particles in Λ, all with speeds at least u, then

$$\begin{aligned} (\mu P^t)\left(\left(\bigcup_{J \leqslant n \leqslant N} X_n \right) \setminus K \right) &\leqslant E(N_{at}(\Lambda \times \{v \,|\, |v| < u\})) \\ &< \epsilon/2 \end{aligned} \tag{3.9}$$

(3.8) and (3.9) yield the theorem.

Theorem 3.6. The measure μ_a^t satisfies $\mu_a^t \ll \lambda$, and $d\mu_a^t/d\lambda$ is bounded, uniformly in t, on each X_n.

Proof. This is an immediate consequence of Lemma 3.4, since for $A \subset X_n$,

$$\mu_a^t(A) \leqslant E\big(N_{at}(\hat{A})\big) \leqslant c_n\lambda_n(\hat{A}) \leqslant c_n\lambda(A)$$

where $\hat{A} = \{(q,v) \in Q_n \times V^n \,|\, (i,q,v) \in A \text{ for some } i\}$.

4. THE INVARIANT MEASURE

We consider an initial measure μ on X and construct an invariant measure ν as a Cesaro limit of the measures $\mu P^t = \mu_a^t + \mu_b^t$. The decay of μ_b^t (Corollary 3.3), the uniform absolute continuity of μ_a^t (Theorem 3.6), and "tightness" (Theorem 3.5) yield a construction which avoids technicalities arising from the lack of continuity of P^t.

Theorem 4.1. There exists a P^t-invariant measure ν on X, absolutely continuous with respect to λ.

Proof. Let μ be a Borel measure on X, write $\mu P^t = \mu_a^t + \mu_b^t$ as usual, and define

$$\nu_\alpha^t = t^{-1}\int_0^t \mu_\alpha^s\,ds, \qquad \alpha = a, b$$

$$\nu^t = \nu_a^t + \nu_b^t = t^{-1}\int_0^t \mu P^s\,ds$$

Then Corollary 3.3 implies that

$$\lim_{t\to\infty} \|\nu_b^t\| = 0 \tag{4.1}$$

By Theorem 3.5 we may choose a sequence $\{K_m\}$ of bounded sets with $K_m \nearrow X$ and $\mu P^t(K_m) \geqslant 1 - 1/m$, and hence $\nu^t(K_m) \geqslant 1 - 1/m$, for all t. (Here in fact we may take $K_m = \bigcup_{J\leqslant n\leqslant N_m} X_n$ for appropriate N_m.) From (4.1) we have

$$1 - \epsilon_m^t \leqslant \nu_a^t(K_m) \leqslant 1 \tag{4.2}$$

with $\epsilon_m^t \to 1/m$ as $t \to \infty$.

Now by Theorem 3.6, $d\nu_a^t(x) = g^t(x)\,d\lambda(x)$, where $|g^t(x)| \leqslant c_{K_m}$ uniformly in $t \geqslant 0$, $x \in K_m$. The closed unit ball in $L^\infty[K_m;\lambda]$ is weak* compact, so that by a diagonal argument we may produce a sequence $\{t_j\}$ and a Borel function g on X such that for any m and any $f \in L^1[K_m;\lambda]$,

$$\int_{K_m} g^{t_j} f\,d\lambda \xrightarrow[j\to\infty]{} \int_{K_m} gf\,d\lambda \tag{4.3}$$

We define ν by $d\nu = g\,d\lambda$. Applying (4.3) with $f = 1$, and using (4.2), yields $1 - 1/m \leqslant \nu(K_m) \leqslant 1$, so that ν is a probability measure on X.

Now let f be any bounded Borel function on X. We have, for any m,

$$\left|\int_X f\,d\nu^{t_j} - \int_X f\,d\nu\right| \leqslant \int_{X\setminus K_m} |f|\,(d\nu^{t_j} + d\nu) + \int_{K_m} |f|\,d\nu_b^{t_j} + \left|\int_{K_m} f\left(d\nu_a^{t_j} - d\nu\right)\right|$$

Thus from (4.1), (4.2), and (4.3)

$$\lim_{j\to\infty}\left|\int_X f\,d\nu^{t_j} - \int_X f\,d\nu\right| \leqslant \frac{2}{m}$$

and since m is arbitrary

$$\int_X f\,d\nu^{t_j} \xrightarrow[j\to\infty]{} \int_X f\,d\nu \tag{4.4}$$

Finally we check the invariance of ν. For any bounded Borel function f, from (4.4),

$$\begin{aligned}\int_X (P^t f - f)\,d\nu &= \lim_{j\to\infty} \int_X (P^t f - f)\,d\nu^{t_j}\\ &= \lim_{j\to\infty} t_j^{-1}\left[\int_{t_j}^{t_j+t}\left(\int_X P^s f\,d\mu\right)ds - \int_0^t\left(\int_X P^s f\,d\mu\right)ds\right]\\ &= 0\end{aligned}$$

We next show that the invariant measure is unique. In the proof of the key intermediate results, Lemma 4.2, we must control the speeds of particles which enter Λ from Λ_+. For this purpose let $W \subset V$ be a fixed interval which is of the form $W = [4u/5, u]$ for some $u > 0$, and which satisfies $\pi_+(W) > 0$. Then for any nonempty time interval (a,b) and any $j \geqslant 0$, there is by (2.2) a nonzero probability that exactly j pulses enter Λ from Λ_+ during (a,b) with speeds in W. Note also that these pulses, if reflected from the wall at $-L/2$, will return to $L/2$ in the time interval $(a + 2T, b + \frac{5}{2}T)$, where $T = L/u$.

Lemma 4.2. There exists a set $A \subset X$ of positive λ^* measure such that for any initial measure μ of bounded support, and for some constant $\eta > 0$ and integer $n > 0$ depending only on the support of μ, $\mu P^{nT} > \eta\lambda^*$ on A. (Recall that λ^* was the reference measure constructed using only velocities from Λ_+; see Definition 2.3.)

Proof. Let the integer $m \geqslant 1$ be chosen so that $m^{-1}u$ is a lower bound for particle speeds on the support of μ. Let $F_1 \subset \Omega$ be the event that no particles enter Λ during the time interval $[0, 2mT]$, and assume that F_1

occurs. Since each pulse in the initial state must hit each wall during $[0, 2mT]$, precisely J initial pulses will survive in Λ until $2mT$.

Now let F_2 be the event that precisely J particles enter Λ from Λ_+ during each time interval $I_k = ((2m + k - 1)T, (2m + k - \frac{1}{2})T)$, for $1 \leqslant k \leqslant 2m + 1$, all with speeds in W, and that no other particles enter Λ during $[2mT, (4m + 2)T]$. Assume that F_2 occurs. The pulses which enter Λ during I_k cannot escape into Λ_-, so they will return to the wall at $L/2$ during $I_k^* + 2 = ((2m + k + 1)T, (2m + k + 2)T)$. Thus during $I = [(2m + 1)T, (4m + 2)T]$ there will be at least J particles in the region $[-L/2, L/2) \subsetneq \Lambda$, so that no molecule can reach $L/2$. It follows that all pulses which reach $L/2$ during I will escape from Λ. This must include all original pulses and all pulses which entered during $I_1, \ldots, I_{2m}$, so that, at time $(4m + 2)T$, Λ will contain precisely those pulses which entered during I_{2m+1}.

Now let $n = (4m + 2)$ and let μ^* be the measure at time nT conditioned on the occurrence of F_1 and F_2. The above argument shows that $\mu^* = Z^{-1}\lambda^*|_A$, where $A \subset X_J$ is a set which depends only on our choice of the speed u and $Z = \lambda^*(A) \neq 0$. Then the conclusion of the lemma follows with $\eta = Z^{-1}\mathbf{P}(F_1 \cap F_2)$.

Corollary 4.3. The measure ν is the unique Borel measure on X invariant under P^t, or under the discrete process $\tilde{\mathbf{P}}^m$, where $\tilde{\mathbf{P}} = P^{kT}$ for some integer $k \geqslant 1$.

Proof. Lemma 4.2 implies that no two invariant measures are mutually singular, from which uniqueness follows.

Now, given any Borel measure μ on X, we write $\mu = \mu_{\text{abs}} + \mu_{\text{sing}}$ as the Lebesgue decomposition with respect to ν. Note that

$$\mu_{\text{abs}} P^t \ll \nu \tag{4.5}$$

for any $t > 0$, that by Lemma 4.2,

$$\lambda^*|_A \ll \nu \tag{4.6}$$

and that for any μ and sufficiently large n,

$$(\mu P^{nT})_{\text{abs}} \neq 0 \tag{4.7}$$

Corollary 4.4. For any initial Borel measure μ, $\|(\mu, P^t)_{\text{sing}}\| \to 0$ as $t \to \infty$.

Proof. From (4.5), $\|(\mu P^t)_{\text{sing}}\| \searrow a$ for some a; suppose $a > 0$. By Theorem 3.5 we may choose a bounded $K \subset X$ with $\|\mu P^t|_{X \setminus K}\| < a/2$ for all t, so that $\|(\mu P^t)_{\text{sing}}|_K\| > a/2$. Let η and n be the constants of Lemma

4.2 for this K, let $Z = \lambda^*(A)$, and choose t with

$$\|(\mu P^t)_{\text{sing}}\| < a(1 + \eta Z/2)$$

Then from (4.5), (4.6), and (4.7),

$$\begin{aligned}
\|(\mu P^{t+nT})_{\text{sing}}\| &= \left\|\left[(\mu P^t)_{\text{sing}} P^{nT}\right]_{\text{sing}}\right\| \\
&= \left\|\left[(\mu P^t)_{\text{sing}}|_K P^{nT}\right]_{\text{sing}}\right\| + \left\|\left[(\mu P^t)_{\text{sing}}|_{X\setminus K} P^{nT}\right]_{\text{sing}}\right\| \\
&\leqslant \|(\mu P^t)_{\text{sing}}|_K\|(1 - \eta Z) + \|(\mu P^t)_{\text{sing}}|_{X\setminus K}\| \\
&= \|(\mu P^t)_{\text{sing}}\| - \eta Z \|(\mu P^t)_{\text{sing}}|_K\| \\
&< a\left(1 + \frac{\eta Z}{2}\right) - \frac{\eta Z a}{2} = a
\end{aligned}$$

This contradiction proves the corollary.

Theorem 4.5. For any initial measure μ, $\|\mu P^t - \nu\| \to 0$ as $t \to \infty$.

Proof. Since for $t = nT + \delta$,

$$\|\mu P^t - \nu\| = \|(\mu P^{nT} - \nu)P^\delta\| \leqslant \|\mu P^{nT} - \nu\|$$

it suffices to prove that $\|\mu \tilde{P}^n - \nu\| \to 0$ as $n \to \infty$, where $\tilde{P} = P^T$. From (4.7) and Corollary 4.3, the Markov chain with transition probability $\tilde{P}$ is an ergodic Harris chain[6,7]; it is aperiodic since the chain with transition probability $\tilde{P}^k$ is ergodic for every positive integer k. Hence[6,7] for ν-almost every y, $\|\delta_y \tilde{P}^n - \nu\| \to 0$ as $n \to \infty$. Write $\bar{\mu}^n = \|(\mu\tilde{P}^n)_{\text{abs}}\|^{-1}(\mu\tilde{P}^n)_{\text{abs}}$. Now given $\epsilon > 0$, we can by Corollary 4.4 find an N such that $\|\mu\tilde{P}^N - \bar{\mu}^N\| < \epsilon$, and hence for $n = N + m$,

$$\|\mu\tilde{P}^n - \bar{\mu}^N\tilde{P}^m\| = \|(\mu\tilde{P}^N - \bar{\mu}^N)\tilde{P}^m\| < \epsilon \tag{4.8}$$

But also

$$\|\bar{\mu}^N\tilde{P}^m - \nu\| = \left\|\int_X (\delta_y \tilde{P}^m - \nu)\, d\bar{\mu}^N(y)\right\| \leqslant \int_X \|\delta_y\tilde{P}^m - \nu\|\, d\bar{\mu}^N(y)$$

which vanishes as $m \to \infty$ by the Lebesgue dominated convergence theorem. This, together with (4.8), proves the theorem.

Remark 4.6. The existence of the invariant measure ν for the system in Λ immediately implies the existence of an invariant probability measure for the entire infinite system, including the reservoirs. Let $\hat{\Omega}$ denote the set of all reservoir states; points of $\hat{\Omega}$ describe particles which are moving away from Λ as well as those which are moving toward Λ (zero-velocity particles are again excluded). We obtain an invariant probability measure on $X \times \hat{\Omega}$

as follows. Let $\mathscr{X} = \{\mathbf{x} = \{x_t\}_{t=-\infty}^{\infty}\}$ be the path space for the process P^t, and let $\mathbf{P}_\nu$ be the measure on $\mathscr{X}$ arising from ν (at time 0). $\mathbf{P}_\nu$ is stationary (invariant under time translation). Consider the natural map $\phi: \mathscr{X} \to X \times \hat{\Omega}$ carrying $\mathbf{x}$ to the state $(x, \hat{\omega})$ at time zero of the entire system whose evolution, observed in Λ, yields $\mathbf{x}$. Since ϕ carries time translation on $\mathscr{X}$ to the time evolution on $X \times \hat{\Omega}$, the probability measure $\mathbf{P}_\nu \cdot \phi^{-1}$ on $X \times \hat{\Omega}$ induced by ϕ is invariant. This measure agrees with $\nu \times \mathbf{P}$ on $X \times \Omega$, and is uniquely determined by this constraint.

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